Math 10A with Professor Stankova
Worksheet, Discussion #10; Monday, 9/18/2017
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Related Rates

Example

1. A circle's area is expanding at a constant rate of $5m^2/s$. How fast is its radius changing when its area is $100\pi m^2$?

> **Solution:** The area of a circle is given by $A = \pi r^2$. Taking the derivative, we have that $A' = 2\pi r r'$. Now we plug in the values that we are give. We know that the area is increasing at a constant rate of 5 so $A' = 5$ and when $A = 100\pi$, we know that $r = \sqrt{100} = 10$. So $5 = 2\pi(10)r'$ so $r' = \frac{5}{20\pi} = \frac{1}{4\pi} m/s$.

2. A spherical meteor is hurtling towards Earth. The angle of how much of the sky it takes up is changing at $1rad/hr$. If we measure the radius of the meteor to be $100m$, how fast is it hurtling towards us when it takes up half of the sky?

> **Solution:** If the meteor is a distance d away and has a radius r, then the sine of half the angle the meteor takes of the sky is r/d. Let θ be the angle of how much the sky the meteor takes up. Then $\sin(\theta/2) = r/d$. Taking the derivative and noting at r is constant, we have that
>
> $$\frac{\cos(\theta/2)\theta'}{2} = \frac{-rd'}{d^2}.$$
>
> When the meteor takes up half the sky, we have that $\theta = \pi/2$ and hence we have that $\sin(\pi/4) = \frac{100}{d}$ so $d = 100\sqrt{2}$. Plugging this all into the equation, we have that
>
> $$\frac{(\sqrt{2}/2)\cdot 1}{2} = \frac{-100 \cdot d'}{(100\sqrt{2})^2}$$
>
> $$\implies d' = -50\sqrt{2}.$$
>
> Therefore, the meteor is hurtling towards us at $50\sqrt{2} m/s$.

Problems

3. A ball of light is falling at a constant rate of $1m/s$. A man who is $2m$ tall is standing $10m$ away. How fast is the length of his shadow changing when the ball is at a height of $4m$?

> **Solution:** If the ball is at height d, then drawing a picture tells us that the height of the shadow satisfies the relation that $\frac{h}{2} = \frac{h+10}{d}$ so $dh = 2h + 20$. Taking the derivative of both sides gives $d'h + dh' = 2h'$. We are given that $d' = -1$ and at $d = 4$, solving for h gives $h = 10$ so
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> $$-10 + 4h' = 2h' \implies h' = 5.$$
>
> So the shadow is increasing at $5m/s$.

4. A conical cup that is $6cm$ wide at the top and $5cm$ tall is filled with water is punctured at the bottom and water is coming out at a rate of $10^{-6}m^3/s$. Initially, the cup is filled How fast is the height of the water changing when the height is $2cm$?

> **Solution:** If the height of the water is h, then the radius of the cone formed by the water would be $3/5h$ and so the volume of the water cone is $V = \pi/3(3/5h)^2 \cdot h = \dfrac{3\pi h^3}{25}$. Taking the derivative of both sides gives
>
> $$V' = \frac{9\pi h^2 h'}{25}$$
>
> and plugging in -10^{-6} for V' and $2 \cdot 10^{-2}$ for h gives
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> $$-10^{-6} = \frac{9\pi 4 \cdot 10^{-4} h'}{25} \implies h' = \frac{-1}{144\pi} m/s.$$

5. A lamppost is $5m$ tall. A woman who is $2m$ tall is walking away from it at a constant rate of $10cm/s$. When she is $2m$ away from the lamppost, how fast is her shadow length changing?

> **Solution:** Using similar triangles, if the woman is at a distance d from the lamppost and the shadow height is h, then
>
> $$\frac{h}{2} = \frac{h+d}{5} \implies 2d = 3h.$$
>
> Taking the derivative, we have that $2d' = 3h'$ and $d' = 10cm/s$ so $h' = \frac{20}{3}cm/s$.

6. Sand is being dumped in a conical pile whose width and height always remain the same. If the sand is being dumped in at a rate of $2m^3/hr$, how fast is the height of the sand changing when the pile is $10cm$ tall?

Solution: Let the height of the pile be h. Then the radius of the pile is $r = \frac{h}{2}$ and the volume of the pile is $V = \frac{\pi r^2 h}{3} = \frac{\pi h^3}{12}$. Taking the derivative gives $V' = \frac{\pi}{4}h^2h'$. Now we plug in 2 for V' and 10^{-1} for h to get $h' = \frac{800}{\pi}m/hr = \frac{800}{3600\pi}m/s = \frac{2}{9\pi}m/s$.

7. A kite is flying at a current altitude of $100m$. The kite slowly flies further and further away as the string length increases at a rate of $3cm/s$. Assuming the altitude does not change, how fast horizontally is the kite moving when the angle the string forms with the ground is $\pi/6$?

Solution: Let ℓ be the length of the rope, and d how far horizontally the kite is flying. Then $\ell^2 = 100^2 + d^2$. Taking the derivative gives $2\ell\ell' = 2dd'$. When the angle the string forms with the ground is $\pi/6$, we calculate that $\ell = 200$ and $d = 100\sqrt{3}$ so $d' = \frac{200\cdot 3\cdot 10^{-2}}{100\sqrt{3}} = 2\sqrt{3}\cdot 10^{-2}m/s$ or $2\sqrt{3}cm/s$.

8. A ladder $5m$ tall is lying against a wall. The bottom of the ladder is pulled out at a rate of $10cm/s$. How fast is the area of the triangle formed by the ladder, wall, and floor changing when the bottom of the ladder is $3m$ away from the wall?

Solution: Let d be how far the bottom of the ladder is away from wall. Then the area of the triangle formed is $\frac{1}{2}\cdot d\cdot\sqrt{25-d^2} = A$. Squaring both sides gives $4A^2 = d^2(25-d^2)$. Now we can take the derivative to get that $8AA' = 2dd'(25 - d^2) + d^2(-2dd')$. When $d = 3$, the area is $\frac{1}{2}\cdot 3\cdot 4 = 6$ and so

$$8\cdot 6\cdot A' = 2\cdot 3\cdot d'(16) + 9(-6d') \implies 48A' = 42d'.$$

Since $d' = 10^{-1}m/s$, we have that $A' = \frac{7}{80}m/s$.

9. A conical volcano is $100m$ tall and the base has a radius of $50m$. It is filling with lava at a rate of $\pi m^3/s$. At what rate is the height of the lava rising with it is $50m$ tall?

Solution: Let h be the height of the lava. The we can calculate the volume of the truncated cone by taking the total area and subtracting the missing top cone. The top cone has a height of $100-h$ and radius of $(100-h)/2$. Thus the volume of the lava is

$$V(h) = \frac{\pi\cdot 50^2\cdot 100}{3} - \frac{\pi\cdot(100-h)^2\cdot(100-h)}{2^2\cdot 3}.$$

Taking the derivative, we get that

$$\frac{dV}{dt} = -\frac{\pi(100-h)^2(-h')}{4}.$$

Since $V' = \pi$, we have that $h' = \frac{4}{50^2} = \frac{1}{625}$.

Optimization

Example

10. Suppose you are trying to make a rectangular fence for your yard. You only have $100m$ of fence but luckily your house borders a straight river, so one side of your rectangular yard will be bordered by a river. What is the largest area yard you can enclose?

Solution: Let s be the side length of the yard that is perpendicular to the river. Then the side length of the yard that is parallel to the river is $(100-2s)$ and the area of the yard is $A(s) = s(100-2s)$. Taking the derivative gives $A'(s) = 100-4s$. So $A' = 0$ when $s = 25$ and note that $A''(s) = -4 < 0$ so this means that $A(25)$ is a local maximum. The largest area is $A(25) = 25(50) = 1250m^2$.

11. What is the closest point to $(0,2)$ on the graph $y = x^2+1$.

Solution: Given a point (x, y) on the curve, we want to minimize the distance $\sqrt{x^2+(y-2)^2}$ but note that this is the same as minimizing $x^2+(y-2)^2$. Now we plug in $y = x^2+1$ so we minimize $x^2+(x^2-1)^2$. Taking the derivative and setting it equal to 0, we have that $2x+2(x^2-1)(2x) = 0$ so $4x^3-2x = 0$ and $2x(2x^2-1) = 0$. The solutions are $x = 0$ and $x = \pm\frac{\sqrt{2}}{2}$. The second derivative is $12x^2-2$ and so $x = 0$ is a local maximum but $x = \pm\frac{\sqrt{2}}{2}$ are local minimums. Thus, there are two points closest which are $(\pm\sqrt{2}/2, 3/2)$.

Problems

12. (4.2, 38) When you cough, the radius of your windpipe decreases and affects the speed of the air through it. If r is the radius of the windpipe, then the speed of the air is $S(r) = ar^2(r_0-r)$ where a, r_0 are constants. Find the radius r for which the speed is the greatest.

Solution: We take the derivative and set it equal to 0 to get $2ar_0r - 3ar^2 = 0$ so $r = 0$ or $r = \frac{2r_0}{3}$. Taking the second derivative, we get $2ar_0 - 6ar$. Thus, the second derivative is positive for $r = 0$ and negative when $r = \frac{2r_0}{3}$ meaning that the second value is a local maximum. So the radius is $\frac{2r_0}{3}$.

13. You want to construct a cylindrical container that contains $100\pi m^3$ of water. What should the dimensions of the container be if you want to minimize the total surface area?

Solution: The surface area is $S(r, h) = 2(\pi r^2) + 2\pi rh$. The volume of the container is $V = 100\pi = \pi r^2 h$. So $h = \frac{100}{r^2}$ and so $S(r) = 2\pi r^2 + \frac{200\pi}{r}$. Taking the derivative and setting it equal to 0 gives $4\pi r - \frac{200\pi}{r^2} = 0$ so $r^3 = 50$ so $r = \sqrt[3]{50}$.

14. An airline is selling tickets for \$200 each and sells 50 per plane. For every \$10 they decrease the price, they sell 10 more tickets. The plane can hold a maximum of 100 passengers. At what price should they sell their tickets for maximum revenue?

Solution: Let x be the amount they decrease the price. Then at a price of $200 - x$ each, they sell $50+x$ tickets. So the total revenue is $R(x) = (200-x)(50+x)$. Taking the derivative, we get $R'(x) = 150 - 2x$. Setting the derivative to 0, we get that $x = 75$ so we should sell $50 + 75 = 125$ tickets. But since the plane has a maximum of 100 passengers and $R'(x)$ for all $50 + x < 125$, this tells us that $x = 50$ is the maximum on the domain of x which is $\{x : x \leq 50\}$. So they should set a price of $200 - 50 = \$150$.

15. Find the rectangle of largest area whose diagonal is of length L.

Solution: Let one of the side lengths of the rectangle by s, then finding the largest area is the same as finding the largest area squared which is $s^2(L^2 - s^2)$. Taking the derivative and setting it equal to 0 gives $2L^2s - 4s^3 = 0$ so $s = 0$ or $s = \frac{L}{\sqrt{2}}$. At $s = 0$, the second derivative is positive and at $s = L/\sqrt{2}$, the second derivative is negative which tells us that $s = L/\sqrt{2}$ gives us the largest area. The other side length is $\sqrt{L^2 - L^2/2} = L/\sqrt{2} = s$ so the largest area is achieved with a square.

16. Find the area of the smallest triangle formed by the x axis, y axis, and a line that goes through the point $(4, 2)$.

Solution: Suppose that the line goes through the point $(0, y_0)$. Then, the slope of the line is $\frac{2-y_0}{4}$ and is described by the line $y - y_0 = \frac{2-y_0}{4}x$. The x intercept is when $y = 0$ or when $x = \frac{4y_0}{y_0-2}$. Thus, the area of the triangle is

$$A(y_0) = \frac{1}{2} \cdot y_0 \cdot \frac{4y_0}{y_0 - 2} = \frac{2y_0^2}{y_0 - 2}.$$

Setting the derivative equal to zero gives $A'(y) = \frac{2y(y-4)}{(y-2)^2}$ so the two solutions are $y = 0$ and $y = 4$. The second derivative is $\frac{16}{(y-2)^3}$ and so $y_0 = 0$ is a local maximum and $y_0 = 4$ is a local maximum. So the area is $\frac{2 \cdot 4^2}{4-2} = 16$.

17. Find the largest rectangle that can be inscribed into a semicircle of radius 1 so that one side of the rectangle is part of the diameter of the semicircle.

Solution: Let the height of the rectangle be h. Then the other side of the rectangle must be $2\sqrt{1-h^2}$. So we want to maximize $2h\sqrt{1-h^2}$, which is the same as maximizing its square $4h^2(1-h^2)$. Setting the derivative equal to 0 gives $8h - 16h^3 = 0$ so $h = 1/\sqrt{2}$. The area is $2/\sqrt{2} \cdot 1/\sqrt{2} = 1$.

18. Suppose you only have $1m$ of wire. You are to construct a circle and a square. What is the maximum and minimum total area of the circle and square?

Solution: Let s be the side length of the square and r be the radius of the circle. Then $4s + 2\pi r = 1$ so $r = \frac{1-4s}{2\pi}$. So the total area is

$$A(s) = s^2 + \frac{\pi(1-4s)^2}{4\pi^2}.$$

Setting the derivative equal to 0 gives $s = \frac{1}{\pi+4}$ and the second derivative is $2 + \frac{8}{\pi}$ which is always positive. Thus, $s = \frac{1}{\pi+4}$ is a local minimum and $A = \frac{1}{16+4\pi}$ is the minimum area. The domain of s is $[0, 1/4]$ so the other critical points are the end point. We have that $A(0) = 1/4\pi$ and $A(1/4) = 1/16$ so the maximum area is $1/4\pi$ which occurs at $s = 0$ so we only make a circle.

Tricky Limits

Problems

Solve all of the following questions without using L'Hopital's rule.

19. Find $\lim_{a\to 2} \frac{a^{2017} - 2^{2017}}{a-2}$.

Solution: Letting $f(x) = x^{2017}$, we recognize this as $\lim_{a\to 2} \frac{f(a) - f(2)}{a-2} = f'(2) = 2017 \cdot 2^{2016}$.

20. Find $\lim_{x\to 1} \frac{e^{3x} - e^3}{x^2 - 1}$.

Solution: We can factor the bottom as $(x-1)(x+1)$. Letting $f(x) = e^{3x}$, we recognize this derivative as

$$\lim_{x\to 1} \frac{e^{3x} - e^3}{x^2 - 1} = \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} \cdot \frac{1}{x+1} = \frac{f'(1)}{2} = \frac{3e}{2}.$$

21. Find $\lim_{x\to 1} \frac{e^{\sqrt{x}} - e}{x^2 - 3x + 2}$.

Solution: Let $f(x) = e^{\sqrt{x}}$ so that $f'(x) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$. Then we can factor the bottom as $(x-1)(x-2)$ and the limit is

$$\lim_{x\to 1} \frac{e^{\sqrt{x}} - e}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} \cdot \frac{1}{x-2} = \frac{f'(1)}{-1} = \frac{-e}{2}.$$

22. Find $\lim_{x\to 0} \frac{\cos x - 1}{x^2 + x}$.

Solution: Let $f(x) = \cos x$. Then the limit is

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2 + x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \cdot \frac{1}{x+1} = \frac{f'(0)}{1} = 0.$$

23. Find $\lim_{x\to 2} \frac{x^2 - 4}{\sqrt{x} - \sqrt{4-x}}$.

Solution: We multiply the top and bottom by the conjugate to get that

$$\lim_{x\to 2}\frac{x^2-4}{\sqrt{x}-\sqrt{4-x}} = \lim_{x\to 2}\frac{(x^2-4)(\sqrt{x}+\sqrt{4-x})}{x-(4-x)} = \lim_{x\to 2}\frac{(x-2)(x+2)(\sqrt{x}+\sqrt{4-x})}{2(x-2)}$$

$$= \lim_{x\to 2}\frac{(x+2)(\sqrt{x}+\sqrt{4-x})}{2} = \frac{4(\sqrt{2}+\sqrt{2})}{2} = 4\sqrt{2}.$$

24. Find $\lim_{x\to\infty}\sqrt{x^2-4x+1}-(x+3)$.

Solution: Multiplying by the conjugate gives

$$\lim_{x\to\infty}\frac{x^2-4x+1-(x+3)^2}{\sqrt{x^2-4x+1}+(x+3)} = \lim_{x\to\infty}\frac{-10x-8}{\sqrt{x^2-4x+1}+(x+3)}.$$

Now dividing the top and bottom by x gives

$$= \lim_{x\to\infty}\frac{-10-8/x}{\sqrt{1-4/x+1/x^2}+(1+3/x)} = \frac{-10}{1+1} = -5.$$